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LETTER TO THE EDITOR

Staircase polygons, scaling functions and asymmetric compact directed percolation

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Abstract

The scaling function for compact directed percolation on a square lattice is investigated for the asymmetric case where two parameters control the critical behaviour. A simple representation for the area–perimeter generating function for staircase polygons is found, which can be recast as a non-linear functional equation. From this, the exact scaling function is extracted. In the process, the most concise derivations to date are given for the exact low order cluster moments.

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Compact directed percolation (CDP) on a square lattice is an exactly solved problem in the sense that the low order cluster moments are known explicitly [1, 2]. This letter is concerned with finding the exact scaling function that determines the asymptotic behaviour of these moments, using techniques that might also be applicable to more difficult (as yet unsolved) problems [3–6]. In particular, it is shown how to handle the full, asymmetric case, which has the specific technical feature that two parameters rather than one control the critical behaviour. To make progress, a mapping between CDP clusters and staircase polygons is utilized. First, a particularly simple continued fraction representation for the area–perimeter generating function of staircase polygons is derived. This is then re-written as a non-linear functional equation, from which exact expressions for the mean cluster perimeter length, width, height and area (number of occupied sites) are obtained almost trivially, the simplest such derivations known. Second, the asymptotic (scaling) behaviour of the generating function is extracted directly from the functional equation. This exact scaling function correctly predicts the divergence of the cluster moments at every point along a critical line in the relevant parameter space.

The correspondence between CDP clusters and staircase polygons is well known [7, 8], so it suffices to summarize the key ideas (figure 1). Let $G(x, y, z)$ be the area–perimeter

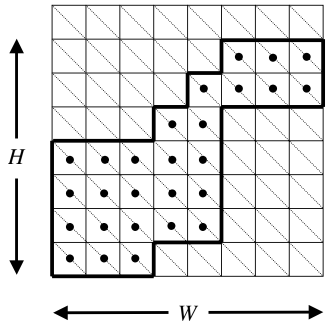

Figure 1. A typical CDP cluster with 27 occupied sites (circles), together with its associated staircase polygon (solid line). The perimeter length is 30, the width is 8 and the height is 7. The activity of this particular polygon is $x^{16}\, y^{14}\, z^{27}$; the probabilistic weight of the cluster is ${p_1}^7\, {p_2}^6\, {q_1}^7\, {q_2}^8 = (p_1q_2)^8\,(p_2q_1)^7/p_1p_2$.

generating function for such polygons, where x and y are the horizontal and vertical perimeter activities, and z is the area activity. This function is singular in the twin limits $x + y \to 1^-$ and $z \to 1^-$ [7, 9], and this governs the nature of the phase transition in CDP. Usually, the analysis is restricted to the symmetric case, $x = y$, but no such restriction is made here. If the CDP (boundary site) occupation probabilities are p_1 and p_2, the cluster moments of interest are given in terms of $G(x, y, z)$ by

$$Q = \frac{1}{p_1 p_2}\, G\Big|_{\substack{x=\sqrt{p_1q_2},\, y=\sqrt{p_2q_1}\\ z=1}} \tag{1}$$

$$L = \frac{Q^{-1}}{p_1 p_2}\left(x\frac{\partial G}{\partial x} + y\frac{\partial G}{\partial y}\right)\Bigg|_{\substack{x=\sqrt{p_1q_2},\, y=\sqrt{p_2q_1}\\ z=1}} \tag{2}$$

$$S = \frac{Q^{-1}}{p_1 p_2}\left(\frac{\partial G}{\partial z}\right)\Bigg|_{\substack{x=\sqrt{p_1q_2},\, y=\sqrt{p_2q_1}\\ z=1}}\,. \tag{3}$$

Here, Q is the probability that a given cluster is finite, L and S are the mean cluster perimeter length and area respectively (*given* that the clusters are finite), and $q_1 = 1 - p_1$ and $q_2 = 1 - p_2$. It is assumed that the initial site is occupied with probability 1. Interpreting (2), $L \equiv 2W + 2H$ with W, H denoting the mean cluster width, height (see figure 1). These expressions are valid either side of the transition.

Several different methods for finding $G(x, y, z)$ are known, leading to quite distinct representations [3, 10–12]. The derivation below, apparently new, is simple and concise. Begin by noting that the number of squares (sites) within the polygon (cluster) which lie on a given diagonal can only stay the same, or increase or decrease by one, as one moves to an adjacent diagonal (see figure 1). Consider, therefore, a sequence of generating functions $g_n(x, y, z)$ obeying the following recursion relation,

$$g_n = xyz^{n-1}g_{n-1} + (x^2 + y^2)z^n g_n + xyz^{n+1}g_{n+1} \tag{4}$$

where $g_0 \equiv 1$. The function g_n generates (a diagonal line at a time) partial staircase structures seeded from (but excluding) an initial diagonal line of n sites. Staircase polygons are seeded from a single site such that $G(x, y, z) = xyzg_1$, as may be verified by direct iteration. When $z = 1$, (4) is trivial to solve. When $z \neq 1$, the key step is to recognize a subtle simplification.

Define $K_n \equiv zg_n/g_{n-1}$ (so that $G(x, y, z) = xyK_1$, since $g_0 \equiv 1$). After some straightforward algebra, it follows that

$$K_n = \frac{xyz^n}{1 - (x^2 + y^2)z^n - xyz^n K_{n+1}}. \tag{5}$$

This is the basis of an elegant continued fraction representation for $G(x, y, z)$,

$$G(x, y, z) = \cfrac{x^2y^2z}{1 - (x^2 + y^2)z - \cfrac{x^2y^2z^3}{1-(x^2+y^2)z^2 - \cfrac{x^2y^2z^5}{1-(x^2+y^2)z^3 - \cfrac{x^2y^2z^7}{\cdots}}}}. \tag{6}$$

Setting $x = y$ and expanding provides an efficient method for enumerating clusters by perimeter and area. Upon close inspection, it is apparent that (6) may also be written as follows,

$$G(x, y, z) = \frac{x^2y^2z}{1 - (x^2 + y^2)z - G(x\sqrt{z}, y\sqrt{z}, z)}$$

and this can be rearranged to give

$$G(x, y, z) = x^2y^2z + (x^2 + y^2)zG(x, y, z) + G(x, y, z)G(x\sqrt{z}, y\sqrt{z}, z). \tag{7}$$

This result is stated in [6] (for $x = y$) with the comment that it can be obtained from a result due to Prellberg and Brak [3] after appropriate ‘symmetrization’. The derivation here represents a ‘cleaner’ approach. In words, (7) states that every staircase polygon is either a single square, or a single square followed by a staircase polygon, or a staircase polygon ‘dressed’ by a class of simply related polygon. The ‘diagonal’ perspective used to derive (7) is the essence of Essam’s original treatment [1] of CDP, based on links with the Domany–Kinzel cellular automaton [2]. However, Essam did not solve for $G(x, y, z)$, choosing instead to evaluate the moments by directly manipulating a linear recursion akin to (4).

The exact moments (1)–(3) can now be obtained almost trivially from (5) or from (7), depending upon taste. The conciseness of the derivations makes this worth demonstrating. Setting $z = 1$ in (7) provides an algebraic (quadratic) equation for the perimeter generating function whose solution is

$$G(x, y, 1) = \frac{1 - (x^2 + y^2) - \sqrt{(1 - (x^2 + y^2))^2 - 4x^2y^2}}{2}. \tag{8}$$

This is singular when $x + y = 1$ or, equivalently, when $p_1 + p_2 = 1$. It follows that $Q(p_1 + p_2 < 1) = 1$ and

$$Q(p_1 + p_2 > 1) = \frac{q_1q_2}{p_1p_2} < 1. \tag{9}$$

Considering instead the probability $P_\infty \equiv 1 - Q$ that a given cluster is infinite, it then follows that $P_\infty(p_1 + p_2 < 1) = 0$ and

$$P_\infty(p_1 + p_2 > 1) = \frac{p_1 + p_2 - 1}{p_1p_2} > 0.$$

The line $p_1 + p_2 = 1$ thus marks the critical boundary, at each point of which the critical exponents are the same [1]. For $p_1 + p_2 < 1$,

$$L = 2 + \frac{2}{(1 - (p_1 + p_2))} \tag{10}$$

which diverges with exponent 1. Similarly, $W = (1 - p_2)/(1 - (p_1 + p_2))$ and $H = (1 - p_1)/(1 - (p_1 + p_2))$, with $L = 2(W + H)$; these results for W and H appear to be new. The results for $p_1 + p_2 > 1$ are obtained via the interchange $p_1 \to q_1$ and $p_2 \to q_2$, a duality that was explained in [1]. Evaluating S is equally straightforward. Differentiating the (implicit) functional equation (7) with respect to z and rearranging gives, for $p_1 + p_2 < 1$,

$$S = \frac{1 - (p_1 + p_2) + p_1 p_2}{(1 - (p_1 + p_2))^2} \tag{11}$$

which diverges with exponent 2. Note that, for this special problem, $S = W \times H$. As an alternative method of derivation, starting from (5) a recursion for $K_n' \equiv \partial K_n/\partial z|_{z=1}$ can be derived, namely,

$$xyK_n' = K^2(n + xyK_{n+1}')$$

where $K \equiv K_n(x, y, 1)$ is independent of n. It is easy to solve for xyK_1' by iteration, whereupon it follows that

$$S = \frac{1}{p_1 p_2} \left. \frac{K^2}{(1 - K^2)^2} \right|_{x=\sqrt{p_1 q_2},\, y=\sqrt{p_2 q_1}}.$$

Since $K = (p_1 p_2/q_1 q_2)^{1/2}$, this reduces to (11). This method of derivation is instructive as it highlights (mathematically) the origin of the divergence. The result for $p_1 + p_2 > 1$ is obtained by duality. All the above results are *exact*.

Turning back now to the main theme, general scaling arguments [7, 9] suggest that for $x, y \neq 0$ in the limit $x + y \to 1^-$ and $\varepsilon \equiv (1 - z) \to 0^+$,

$$G(x, y, z) \sim \frac{1 - (x^2 + y^2)}{2} + \varepsilon^\theta F\left(\frac{1 - (x + y)}{\varepsilon^\varphi}\right) \tag{12}$$

where $F(t)$ is a scaling function to be identified. Prellberg [9] studied a q-series representation for $G(x, y, z)$ and was able to extract (in a difficult and sophisticated analysis) the asymptotic behaviour in the limit $\varepsilon \to 0$ $(0 < x, y < 1)$. Upon the taking the additional limit $x + y \to 1$ the validity of (12) was established and the scaling function found explicitly. The question of generic interest, however, is whether $F(t)$ can be extracted directly from the defining functional equation for $G(x, y, z)$ [3–6]. For the symmetric case $x = y$ the answer is known to be yes (the reader is referred to [5, 6] for details). In extending this work to the more complicated, asymmetric case a number of subtle technical issues have to be faced.

Define a scaling variable $t = (1-(x+y))\varepsilon^{-\varphi}$, so that the 'critical' variable $x+y = 1-t\varepsilon^\varphi$. The limits $x + y \to 1^-$ and $\varepsilon \to 0^+$ are taken with t fixed, and in the following specific sense. Consider a rotated 'co-ordinate' system by introducing the orthogonal variables $v = (x+y)/\sqrt{2}$ and $u = (x - y)/\sqrt{2}$. The limit $x + y \to 1$ corresponds to $v \to 1/\sqrt{2}$. The *choice* is made to take this limit with u (i.e. $x - y$) fixed (see the later comment). Using the $(x, y) \Leftrightarrow (v, u)$ transformation as a guide, the following quantities appearing in (7) have the 'natural' form:

$$x^2 y^2 \equiv \tfrac{1}{4}[v^2 - u^2]^2 = \tfrac{1}{16}[(1 - t\varepsilon^\varphi)^2 - (x - y)^2]^2$$
$$x^2 + y^2 \equiv v^2 + u^2 = \tfrac{1}{2}[(1 - t\varepsilon^\varphi)^2 + (x - y)^2].$$

A differential equation for $F(t)$ is now obtained by expanding (7) in powers of ε and retaining the lowest order, non-trivial terms; the so-called method of dominant balance [5, 6]. A key observation is that

$$\varepsilon^\theta F\left(\frac{1 - (x + y)\sqrt{1 - \varepsilon}}{\varepsilon^\varphi}\right) \sim \varepsilon^\theta F(t) + \frac{1}{2}\varepsilon^{1+\theta-\varphi}\frac{\mathrm{d}F(t)}{\mathrm{d}t}.$$

A non-trivial expression for $F(t)$ is obtained *only* if the crossover exponents $\theta = 1/3$ and $\varphi = 2/3$, whereupon equating powers of $\varepsilon^{2/3}$,

$$F(t)^2 + \frac{(1 - (x - y)^2)}{8}\frac{\mathrm{d}F(t)}{\mathrm{d}t} - \frac{(1 - (x - y)^2)}{2}t = 0. \tag{13}$$

The solution of this non-linear (Ricatti) equation is (cf the results in [6]),

$$F(t) = \frac{(1-(x-y)^2)}{8}\frac{\mathrm{d}}{\mathrm{d}t}\ln \mathrm{Ai}\left(\frac{2^{5/3}}{(1-(x-y)^2)^{1/3}}t\right) \tag{14}$$

where Ai (t) is the Airy function. This is the exact scaling function. It is in agreement with the more general result obtained in [9] in the appropriate limit. In fact, the result in [9] suggests that one can relax the way the limit $x + y \to 1$ is taken to derive (14). Exactly how to carry this out in practice, however, is not clear.

The asymptotic behaviour of the low order moments can be determined from (14). Using standard properties of Airy functions, as $t \to \infty$

$$F(t) \sim -\frac{[1-(x-y)^2]^{1/2}t^{1/2}}{\sqrt{2}} - \frac{[1-(x-y)^2]}{32t}.$$

The limit $x + y \to 1^-$ implies $p_1 + p_2 \to 1$ in the sense of

$$1-(x+y) \sim (1-(p_1+p_2))^2/8p_1p_2$$

and $x - y \sim p_1 - p_2$. Using these results, e.g. for $p_1 + p_2 \to 1^-$, gives

$$L \sim \frac{2}{1-(p_1+p_2)} \qquad W \sim \frac{p_1}{1-(p_1+p_2)} \qquad H \sim \frac{p_2}{1-(p_1+p_2)}$$
$$S \sim \frac{p_1p_2}{(1-(p_1+p_2))^2} \qquad S \sim W \times H.$$

These are correct along the critical line (cf the exact results with $p_1 + p_2 \sim 1$). The recovery of the correct amplitudes is a significant check on the validity of (14).

In conclusion, the exact scaling function governing the phase transition in asymmetric compact directed percolation has been obtained directly from a non-linear functional equation for the staircase polygon generating function. A new derivation has been provided for the latter and, in passing, very concise derivations have been given for the exact low order cluster moments.

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